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# AN INTEGRATION METHOD WITH FITTING CUBIC SPLINE FUNCTIONS TO A NUMERICAL MODEL OF 2ND-ORDER SPACE-TIME DIFFERENTIAL REMAINDER ——FOR AN IDEAL GLOBAL SIMULATION CASE WITH PRIMITIVE ATMOSPHERIC EQUATIONS

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**Abstract:** In this paper, the forecasting equations of a 2nd-order space-time differential remainder are deduced from the Navier-Stokes primitive equations and Eulerian operator by Taylor-series expansion. Here we introduce a cubic spline numerical model (Spline Model for short), which is with a quasi-Lagrangian time-split integration scheme of fitting cubic spline/ bicubic surface to all physical variable fields in the atmospheric equations on spherical discrete latitude-longitude mesh. A new algorithm of "fitting cubic spline—time step integration—fitting cubic spline—……" is developed to determine their first- and 2nd-order derivatives and their upstream points for time discrete integral to the governing equations in Spline Model. And the cubic spline function and its mathematical polarities are also discussed to understand the Spline Model's mathematical foundation of numerical analysis. It is pointed out that the Spline Model has mathematical laws of "convergence" of the cubic spline functions contracting to the original functions as well as its 1st-order and 2nd-order derivatives. The "optimality" of the 2nd-order derivative of the cubic spline functions is optimal approximation to that of the original functions. In addition, a Hermite bicubic patch is equivalent to operate on a grid for a 2nd-order derivative variable field. Besides, the slopes and curvatures of a central difference are identified respectively, with a smoothing coefficient of 1/3, three-point smoothing of that of a cubic spline. Then the slopes and curvatures of a central difference are calculated from the smoothing coefficient 1/3 and three-point smoothing of that of a cubic spline, respectively. Furthermore, a global simulation case of adiabatic, non-frictional and "incompressible" model atmosphere is shown with the quasi-Lagrangian time integration by using a global Spline Model, whose initial condition comes from the NCEP reanalysis data, along with quasi-uniform latitude-longitude grids and the so-called "shallow atmosphere" Navier-Stokes primitive equations in the spherical coordinates. The Spline Model, which adopted the Navier-Stokes primitive equations and quasi-Lagrangian time-split integration scheme, provides an initial ideal case of global atmospheric circulation. In addition, considering the essentially non-linear atmospheric motions, the Spline Model could judge reasonably well simple points of any smoothed variable field according to its fitting spline curvatures that must conform to its physical interpretation.

**Key words:** numerical forecast and numerical simulation; 2nd-order space-time differential remainder numerical model; cubic spline functions; Navier-Stokes primitive equations; quasi-Lagrangian time-split integration scheme; global simulation case

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## 1 INTRODUCTION

Space discretization and time integration are the two sides of a "coin"— dynamic core of a numerical model, which determines physical mechanism, mathematical accuracy and computational error of the atmospheric dynamical system. It also identifies the model physical conservations (such as conservation of mass, energy, etc.), mathematical properties and computational stabilities, holding the atmospheric movements in the model.

In numerical analysis[1-4], cubic spline functions include cubic spline, bi-cubic surface and tri-cubic (3-D cubic) cube, while the cubic spline is the mathematical foundation of the other two. Nowadays, the bicubic surface, which was first suggested to

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conduct aircraft design by Fergusion[5] working in Boeing CO. Ins, is perfectly described by computer graphics.

Cubic spline function comes from Hermite 2nd-order derivative interpolation[1]. Some mathematic laws of the cubic spline, bicubic surface and tricubic cube are as follows. 1) The cubic interpolation function, together with its 1st-order and 2nd-order derivatives, contracts to the original function (Contraction law); 2) its 2nd-order derivative is an optimal approximation of the original function (Optimality law); and 3) there exists periodic cubic spline interpolation. As the Spectral Model was based on some similar laws, a "cubic spline model" (short for Spline Model) should be presented by a 2nd-order differentiable ($C^3$ continuity), "convergence" and "optimal" model with the cubic spline functions fitting to non-linear variable fields in the equations of atmospheric motions. Thus the dynamic core of Spline Model may be a quasi-Lagrangian integration scheme fitting with all variable fields of the cubic spline interpolations. All of them contain 2nd-order spatial derivatives, i.e., their slopes, curvatures and torsions are stationary. And it is easier to solve over-close grids in the Antarctic/Arctic areas as well as the two poles in a global Spline Model by the bicubic surface interpolation.

It is known that the slope ( $\overline{m}_i$ , 1st-order differential quotient) and curvature ($\overline{M}_i$, 2nd-order differential quotient) of the equidistant ($\Delta X$) central difference are equal to ($m_i, M_i$) of the cubic spline, respectively. Both of them need a three-point smoothing and the smoothing coefficient is 1/3, and the formulas are as follows[1]:

$$\overline{m}_i = m_i + \frac{1}{3}(\frac{m_{i+1} - 2m_i + m_{i-1}}{2}) = \frac{P_{i+1} - P_{i-1}}{2\Delta X}$$

$$\overline{M}_i = M_i + \frac{1}{3}(\frac{M_{i+1} - 2M_i + M_{i-1}}{2}) = \frac{P_{i+1} - 2P_i + P_{i-1}}{\Delta X^2}$$

The formula of $\overline{M}_i$ is the foundation of fitting cubic spline. It is called the "chase-after method"[1], which indicates that the format of cubic spline is perfectly consistent with that of 2nd-order central differential quotient.

The quasi-Lagrangian time-split integration scheme with fitting of cubic spline functions is investigated in this work to get a global Spline Model of 2nd–order space-time differential remainder. Its calculation accuracy is higher than the Euler method model of central difference and quasi-Lagrangian method model with bilinear interpolation.

Robert[6] and Robert et al.[7] introduced semi-Lagrangian and semi-implicit numerical integration schemes for the primitive equations to a multilevel model. Bates et al.[8] and Qian et al.[9] considered that Newton's formula of distance, velocity and acceleration and linear interpolation or iterative interpolation calculation with 2nd-level time can be used to calculate the path of quasi-Lagrangian upstream air parcel as well as many other variables, but special space-time discretization was necessary for a scalar or vector field. Purser and Leslie[10] and Noir et al.[11] improved a reduced-order interpolation method, which had a better calculation accuracy and computational efficiency. They applied it to an operational numerical model, since they believed that the calculation of upstream path of an air parcel with the traditional non-linear interpolation, like cubic spline, cost too much computational time. Layton[12] presented a new numerical method of fitting cubic spline to the shallow water equations (SWEs) in spherical coordinates. In her viewpoint, the spatial discretization schemes used in meteorological applications were limited to low-order finite-difference methods, while the spectral method with high-order approximation asked for Legendre transforms, which shows some computational complexity. In her implementation, the SWEs were discretized in a 3rd-time layer with the semi-implicit and leap-frog integral formula, whereas the cubic spline fitting to skipped latitude-longitude grids was used in space. Numerical results demonstrated the stability and accuracy of the new method. Gu et al.[13-15] successfully showed an ideal case in simulating the non-linear, bicubic- surface advections in 120-h integration with time step of 180 s by using a global model with a quasi-Lagrangian integration scheme, which is a set of global quasi-uniform latitude-longitude grids with the sets in high latitudes skipped. The NCEP re-analysis data was treated as an initial model atmosphere. The integration and step values were decided based on the following reason. (1) It was easy to deal with over-close meshes in high latitudes by fitting spline interpolations to grids skipped; and (2) the horizontal wind field in the Antarctic and Arctic poles can be expressed as the 2nd-order derivative of the entire spherical surface in the *z*-coordinates. By using Taylor series expansion and time-to-space derivative transformation to the Eulerian operator, Gu[16, 17] deduced a docking derivation of 4th-order time-to-space differential remainder between the Eulerian scheme and the quasi-Lagrangian scheme, demonstrating the same solution of them, i.e. Eulerian track's slope, curvature and torsion and quasi-Lagrangian path would have the same 2nd-order differential remainders with fitting cubic spline functions. The 4th-order differential remainder also can be calculated, but the number of computation increased exponentially for fitting enormous splines.

## 2 FORECAST EQUATIONS OF 2ND-ORDER SPACE-TIME DIFFERENTIAL

**REMAINDER**

2.1 *Navier-Stokes primitive equations*

On the spherical *z*-coordinates, the adiabatic and non-topography "shallow atmosphere" Navier-Stokes primitive equations are as follows:

$$\frac{d\ln p}{dt}=\frac{-1}{1-\kappa}\nabla\cdot\vec{V} \tag{1}$$

$$\frac{d\ln T}{dt}=\frac{-\kappa}{1-\kappa}\nabla\cdot\vec{V} \tag{2}$$

$$\frac{dq}{dt}=0 \tag{3}$$

$$\frac{du}{dt}=-\mathrm{R}T\frac{\partial\ln p}{\partial x}+fv-\tilde{f}w+(uv\cdot\tan\varphi-uw)/a_0+F_u \tag{4}$$

$$\frac{dv}{dt}=-\mathrm{R}T\frac{\partial\ln p}{\partial y}-fu-(u^2\tan\varphi+vw)/a_0+F_v \tag{5}$$

$$\frac{dw}{dt}=-\mathrm{R}T\frac{\partial\ln p}{\partial z}-g+\tilde{f}u+(u^2+v^2)/a_0+F_w \tag{6}$$

in which $t$ stands for the time, $a_0$ is the average radius of the earth, $\lambda$ and $\phi$ are the longitude and latitude, $r=(a_0+z)$ is the distance between the air parcel and the geocentric point, respectively. Here $\delta x=a_0\cdot\cos\varphi\delta\lambda$ , $\delta y=a_0\cdot\delta\varphi$ , $\delta r=\delta z$ , $f=2\Omega\sin\varphi$ and $\tilde{f}=2\Omega\cos\varphi$ . While $\Omega$ , $(F_u,F_v,F_w)$ , $g$ is for the angular earth rotation velocity, the frictions, and the gravitational acceleration, respectively. Let $\kappa=\mathrm{R/C_p}$ and $R=R_d(1+0.618q)$ in which, $R_d$ , $R$ and $C_p$ are dry, wet specific air constant, and wet air specific heat at constant pressure, respectively. $p$ , $T$ , $q$ , $(u,v,w)$ is pressure, temperature, specific humidity, and wind speeds in order.

Then, all of the forecasting equations (1) - (6) can be written in a general formula ($P$ stands for $p$ , $T$ , $q$ , $u$ , $v$ , $w$ ):

$$\frac{\mathrm{d}P}{\mathrm{d}t}=a \tag{7}$$

in which $a$ stands for the forcing item of a generalized acceleration for every forecasting equation. For the $(u,v,w)$ equations, $a$ is the generalized Newtonian force acting on unit air mass on the rotating Earth, noted as $(a_u,a_v,a_w)$ correspondingly from now on. For the $(p,T)$ equations, it is the acting items of three-dimensional divergence. For the $q$ equation, it stands for the vapor variability of source/sink, which could be zero in dry adiabatic.

2.2 *General forecast equation of 2nd-order space-time differential remainder*

It is known that truth-seeking solution of the Euler operator is

$$\frac{dP(t,x,y,z)}{dt}=\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}+w\frac{\partial P}{\partial z}$$

Thus, the general formula for $P$ of the forecasting equations (1) - (6) becomes

$$\frac{\partial P}{\partial t}=-u\frac{\partial P}{\partial x}-v\frac{\partial P}{\partial y}-w\frac{\partial P}{\partial z}+a \tag{8}$$

The $\Delta t$ Taylor series expansion of the forecast variable $P(t+\Delta t,x,y,z)$ is derived as:

$$P(t+\Delta t,x,y,z)=P(t,x,y,z)+\Delta t\frac{\partial P}{\partial t}+\frac{\Delta^2 t}{2!}\frac{\partial^2 P}{\partial t^2}+...+R_n(t+\Delta t,x,y,z) \tag{9}$$

in which $R_n$ is $\Delta^{n+1}t$ -order differential remainder of magnitude. It is commonly used to replace $O(\Delta^n t)$ in meteorology.

With $N$-order ($N$>4) time-to-space derivatives of Eq. (8) for the Euler operator of $P$ , the time partial derivatives of $P$ in Eq. (9) could be converted to the sum of space partial derivatives, the acceleration terms of $P$ and their derivatives. When omitting all the above 3rd-order partial derivatives of $P$ , however, all partial derivatives of wind velocity $(u,v,w)$ and acceleration $(a_u,a_v,a_w)$ , the general forecast equation of 2nd-order space-time differential remainder is expressed as follows:

$$\begin{aligned}P(t+\Delta t,x,y,z)&\approx P(t,x,y,z)-\Delta x\frac{\partial P}{\partial x}-\Delta y\frac{\partial P}{\partial y}-\Delta z\frac{\partial P}{\partial z}\\&+\frac{\Delta^2 x}{2}\frac{\partial^2 P}{\partial x^2}+\frac{\Delta^2 y}{2}\frac{\partial^2 P}{\partial y^2}+\frac{\Delta^2 z}{2}\frac{\partial^2 P}{\partial z^2}\\&+\Delta x\Delta y\frac{\partial^2 P}{\partial x\partial y}+\Delta x\Delta z\frac{\partial^2 P}{\partial x\partial z}+\Delta y\Delta z\frac{\partial^2 P}{\partial y\partial z}+a\Delta t\\&=\hat{P}(t,x-\Delta x,y-\Delta y,z-\Delta z)+a\Delta t\end{aligned} \tag{10}$$

in which 3-D displacements $(\Delta x,\Delta y,\Delta z)$ have been taken as:

$$\Delta x=u\cdot\Delta t+a_u\frac{\Delta^2 t}{2};$$

$$\Delta y=v\cdot\Delta t+a_v\frac{\Delta^2 t}{2};$$

$$\Delta z=w\cdot\Delta t+a_w\frac{\Delta^2 t}{2} \tag{11}$$

For Eq. (10), $\hat{P}(t, x-\Delta x, y-\Delta y, z-\Delta z)$ is obviously the approximation of $O(\Delta^2 x, \Delta^2 y, \Delta^2 z)$-order differential remainder of magnitude for the 3-D upstream point $P(t, x-\Delta x, y-\Delta y, z-\Delta z)$. Eq. (10) shows that if the quasi-Lagrangian air parcel does not exchange with the outside environment ($a=0$), its forecast value at the Euler point will be equal to the value of an upstream point. However, the 3-D displacement field of all the upstream points should be non-linear paths through each variable field, while they are just 3-D "cubic spline" paths in Eq. (10).

### 2.3 *Equations of 2nd-order space-time differential remainder with cubic spline functions*

It is assumed that the atmospheric motion variable fields ($P$) are all 2nd-order derivatives, and their fitting slopes, curvatures and twists can be obtained by the cubic spline interpolations. In Eq. (10), there could be some approximate values:

$$\frac{\partial P}{\partial x} = P^x, \quad \frac{\partial P}{\partial y} = P^y, \quad \frac{\partial P}{\partial z} = P^z,$$

$$\frac{\partial^2 P}{\partial x^2} = P^{xx}, \quad \frac{\partial^2 P}{\partial y^2} = P^{yy}, \quad \frac{\partial^2 P}{\partial z^2} = P^{zz},$$

$$\frac{\partial^2 P}{\partial x \partial y} = P^{xy}, \quad \frac{\partial^2 P}{\partial x \partial z} = P^{xz}, \quad \frac{\partial^2 P}{\partial y \partial z} = P^{yz}$$

Thus Eq. (10) becomes

$$\begin{aligned} P(t+\Delta t, x, y, z) &\approx P(t, x, y, z) \\ &-\Delta x P^x - \Delta y P^y - \Delta z P^z \\ &+\frac{\Delta^2 x}{2} P^{xx} + \frac{\Delta^2 y}{2} P^{yy} + \frac{\Delta^2 z}{2} P^{zz} \\ &+\Delta x \Delta y P^{xy} + \Delta x \Delta z P^{xz} + \Delta y \Delta z P^{yz} + a\Delta t \\ &\approx \hat{P}(t, x-\Delta x, y-\Delta y, z-\Delta z) + a\Delta t \end{aligned} \quad (12)$$

However, for reducing the huge calculation of fitting cubic spline, a "scale analysis" method can be adopted to simplify the problem. Because the large-scale horizontal motion is much greater than the vertical motion, as $\Delta x >> \Delta z$, and $\Delta y >> \Delta z$. Thus the smaller items in Eq. (12) containing $P^{xz}$ and $P^{yz}$ are omitted, and Eq. (12) becomes

$$\begin{aligned} P(t+\Delta t, x, y, z) &\approx P(t, x, y, z) \\ &-\Delta x P^x - \Delta y P^y - \Delta z P^z \\ &+\frac{\Delta^2 x}{2} P^{xx} + \frac{\Delta^2 y}{2} P^{yy} + \Delta x \Delta y P^{xy} + a\Delta t \\ &+\frac{\Delta^2 x}{2} P^{xx} + \frac{\Delta^2 y}{2} P^{yy} + \Delta x \Delta y P^{xy} + a\Delta t \\ &\approx \dot{P}(t, x-\Delta x, y-\Delta y, z) \\ &-\Delta z \cdot P^z + \frac{\Delta^2 z}{2} P^{zz} + a\Delta t \end{aligned} \quad (13)$$

In Eq. (13), $\dot{P}(t, x-\Delta x, y-\Delta y, z)$ is the approximation of $O(\Delta^2 x, \Delta^2 y)$-order differential remainder in magnitude of the 2-D upstream point $P(t, x-\Delta x, y-\Delta y, z)$. As a result, $\dot{P}$ can be achieved by fitting bicubic surface instead of fitting tricubic cube to get $\hat{P}$.

According to Eqs. (12) and (13), we have $\hat{P} \approx \dot{P} - \Delta z \cdot P^z + \frac{\Delta^2 z}{2} P^{zz}$. It is convenient to get the discrete forecast equations, including the 2nd-order space-time differential remainder

( $O(\Delta^2 t, \Delta^2 x, \Delta^2 y, \Delta^2 z)$

or

$O(\Delta^2 t, \Delta^2 x, \Delta^2 y, \Delta z)$ ) of Eqs. (1) - (6):

$$p^{t+\Delta t} = \hat{p}^t \exp(-\frac{\Delta t}{1-\kappa} \nabla \cdot \vec{V}) \quad (14)$$

$$T^{t+\Delta t} = \hat{T}^t \exp(-\frac{\kappa \Delta t}{1-\kappa} \nabla \cdot \vec{V}) \quad (15)$$

$$q^{t+\Delta t} = \hat{q}^t \quad (16)$$

$$u^{t+\Delta t} = \hat{u}^t + a_u^t \Delta t \quad (17)$$

$$v^{t+\Delta t} = \hat{v}^t + a_v^t \Delta t \quad (18)$$

$$w^{t+\Delta t} = \hat{w}^t + a_w^t \Delta t \quad (19)$$

## 3 HERMITE BICUBIC PATCH AND COONS BICUBIC SURFACE

In numerical analysis, a patch is the simplest mathematical element of fitting curved surface. Set a patch as $\Pi(P)$, in which $P$ stands for $p, T, q, u, v, w$, etc.. On a spherical latitude-longitude mesh, let $x$ ($x \in [0,1]$), $y$ ($y \in [0,1]$) be two independent parameters defined on the latitudinal and meridional directions, respectively. They will make all of the global latitude-longitude meshes become topological rectangular. Then the Hermite bicubic patch is perfectly determined by a matrix of 16 independent vectors:

$$\Pi(P) = \begin{vmatrix} P_{00} & P_{01} & P_{00}^y & P_{01}^y \\ P_{10} & P_{11} & P_{10}^y & P_{11}^y \\ P_{00}^x & P_{01}^x & P_{00}^{xy} & P_{01}^{xy} \\ P_{10}^x & P_{11}^x & P_{10}^{xy} & P_{11}^{xy} \end{vmatrix} \quad (20)$$

In Eq. (20) the subscript of each $P$ stands for the position in its 4 vertices, while the superscript is for the partial derivative of $x$ and/or $y$. In the matrix $P$, the

4 upper-left vectors are known, and we can deduce that the 4 upper-right and 4 lower-left are 1st-order derivatives (slopes), i.e., $P_{00}^{xx}$, $P_{01}^{xx}$, $P_{10}^{xx}$, $P_{11}^{xx}$, $P_{00}^{yy}$, $P_{01}^{yy}$, $P_{10}^{yy}$, and $P_{11}^{yy}$, corresponding to the 8 2nd-order derivatives (curvatures) derived by fitting cubic spline on the set of parameterized *x* or *y* nodes in the *P* field. The former and the latter 8 values depend on each other. The 4 lower-right 2nd-order partial derivatives (twists, $P^{xy}/P^{yx}$) are gained by fitting the spline of known $P^x$ / $P^y$ on the parameterized *y*/*x*, $P^{xy}=P^{yx}$.

The algebraic expression of a parameterized bicubic patch, when projected on a topological rectangular mesh ($x\in[0,1]$, $y\in[0,1]$), is

$$P(x,y)=a_{33}x^3y^3+a_{32}x^3y^2+a_{23}x^2y^3+a_{31}x^3y+a_{13}xy^3+a_{30}x^3+a_{03}y^3 \\ +a_{22}x^2y^2+a_{21}x^2y+a_{12}xy^2+a_{20}x^2+a_{02}y^2+a_{11}xy+a_{10}x+a_{01}y+a_{00} \quad (21)$$

Eq. (21) is equivalent to the matrix $\Pi$ of a Hermite bicubic patch in Eq. (20). Substitute Eq. (21) with 16 known independent vectors (4 values and 12 partial derivatives) in matrix $\Pi$, we will obtain a series of 16 linear equations. Therefore, the 16 algebraic coefficients can be estimated in Eq. (21).

According to the continuity requirements of fitting a bicubic curved surface, a Coons[2] bicubic surface over a limited area or all of the Earth in spherical coordinates is 2nd-order differentiable, i.e., $C^2$ continuity at any (*x* or *y*) directions. The Coons bicubic surface can be formed by a lot of analogous Hermite bicubic patches on the global latitude-longitude meshes, ①slope and ②curvature are two boundary conditions for a cubic spline and a bicubic surface in some limited area:. For instance, the forward/backward differential could be used to get the slope at the two ends of cubic spline.

## 4 QUASI-LAGRANGIAN TIME-SPLIT INTEGRATION SCHEME

Eq. (13) indicates that a global Spline Model can make one appropriate time step integration for the model atmosphere 2nd-order differentiable on the topological rectangular grid with spherical latitude-longitude by fitting bicubic surface in horizontal and cubic spline to each field of *p*, *T*, *q*, $(u,v,w)$ and $(a_u,a_v,a_w)$ in the vertical.

### 4.1 *Navier-Stokes primitive equations*

The main feature of quasi-Lagrangian time integration of the global Spline Model is that it is only needed to calculate a level upstream point and its value in a patch ($\Pi$) on one of the four topological rectangles adjacent to a forecast point. According to Newton's relation of distance, speed and acceleration, we have an upstream point for its horizontal displacements L $(\Delta x,\Delta y)$: $(\Delta x,\Delta y)=-(u\Delta t+\frac{a_u}{2}\Delta^2 t, v\Delta t+\frac{a_v}{2}\Delta^2 t)$. In order to obtain L more accurately, $\Delta t$ is further divided into *N* time slices to make the air parcel path closer to the real wind track. Meanwhile, the computational stability criterion of Courant-Friedrichs-Levy (CFL) must be satisfied. Otherwise the upstream point along the streamline L would locate at some other patches out of the four bicubic patches. Its quadrant and location are determined by its positive/negative sign and size when editing codes, so that we can get the air parcel's predicted value on the patch ($\Pi$) with the known stationary slope, curvature and torsion. In the same way, we obtain the vertical displacement $\Delta z$ of the upstream point and all of its predicted variable values, $\hat{P}\approx\dot{P}-\Delta z\cdot P^z$, in the global Spline Model by fitting cubic spline to every column's variable field (*P*) in the vertical.

### 4.2 *Calculating three-dimensional divergence and time-split integration*

For seeking a global three-dimensional displacement field of all the cubic-motion upstream points, it is necessary to forecast the transportation of atmospheric quality and energy of the "horizontal advection on bicubic surface + vertical convection in cubic spline". The "implicit" three-dimensional divergence field of the upstream points should also be obtained at the same time ($\Delta t$).

Let I, J, K be the number of forecast points in $(x,y,z)$ directions, respectively. For the 3-D displace field, there is $S_{(\mathrm{i,j,k})}=(\Delta x_{\mathrm{i}},\Delta y_{\mathrm{j}},\Delta z_{\mathrm{k}})$ in one step $\Delta t$, in which $\mathrm{i}=0,1,2,......,\mathrm{I}$, $\mathrm{j}=0,1,2,......,\mathrm{J}$, $\mathrm{k}=0,1,2,......,\mathrm{K}$. Actually, cubic spline could be fitted to them in each direction (take forward/ backward difference in spline bounds of the top and bottom), to strike all of their slopes, i.e. $(S^x_{(\mathrm{i,j,k})},S^y_{(\mathrm{i,j,k})},S^z_{(\mathrm{i,j,k})})$. In the "implicit" time, the average 3-D divergence field is:

$$\Delta t\nabla \cdot \vec{V} = \Delta t(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - v\frac{\tan\varphi}{a_0} + \frac{\partial w}{\partial z}) \approx (S^x + S^y - \Delta y_{\mathrm{j}}\frac{\tan\varphi}{a_0} + S^z)^{t+\Delta t} \quad (22)$$

The horizontal displacements of air parcel are mostly attributed from a resultant force, i.e. $(a_u, a_v)$, of the horizontal pressure gradient force and Coriolis force. Its vertical displacement is primarily due to vertical pressure gradient force and the Earth's gravitational force $(a_w)$. As a result, we should take a long time step $(\Delta t)$ to find the horizontal path and a short time step to find the vertical path to get the air parcel's 3-D tracks. We can also further divide the time step ( $\Delta t$ ) into M time slices ( $\delta t$ ), i.e. $\Delta t = \mathrm{M}\delta t$ , ( $\delta t$ is related to the Brunt-Vaisala frequency of gravity wave). $(\Delta x_{\mathrm{i}}, \Delta y_{\mathrm{j}})$ can be obtained by $\Delta t$ while $(\Delta z_{\mathrm{k}})$ can be obtained by $\delta t$ (so called "time-split "). Then Eq. (22) is easily transformed to:

$$\delta t\nabla \cdot \vec{V} \approx (\frac{S^x + S^y}{\mathrm{M}} - \frac{\Delta y_{\mathrm{j}}}{\mathrm{M}}\frac{\tan\varphi}{a_0})^{t+\Delta t} + (S^z)^{t+\delta t} \quad (23)$$

Furthermore, it is not difficult to change the pressure and temperature forecast equations (14) - (15) correspondingly with $\delta t\nabla \cdot \vec{V}$.

Under the condition of hydrostatic equilibrium, we can fit cubic spline to the static equation in each column and perform spatial integral at every moment of $t + \mathrm{m}\delta t$ ( $\mathrm{m} = 1,2,...\mathrm{M}$ ). From the results, all the cubic motion $(\frac{\Delta x}{\mathrm{M}}, \frac{\Delta y}{\mathrm{M}}, \delta z)_{\mathrm{k}}$ and the vertical displacement $(\delta z_{\mathrm{k}})$ of an air parcel in every "time-split" $\delta t$ can be achieved, as well as the static, geopotential height differences at each layer with respect to their *z*-coordinates.. The $(S^z)^{t+\delta t}$ can also be obtained by fitting spline to the $\delta z_{\mathrm{k}}$. In theory, an iterative method must be used to find the height differences $\delta z_{\mathrm{k}}$ as well as its $(S^z)^{t+\delta t}$ and to get the predicted values of the $p^{t+\delta t}$ and $T^{t+\delta t}$ at the same time until any vertical height difference $(\delta z_{\mathrm{K}})$ in top of the column stops changing.

Obviously, one can take the "time-split" integration to reduce the computation load significantly via fitting bicubic surface to a horizontal variable field.

## 5 AN IDEAL SIMULATION CASE

The global Spline Model is characterized by[14, 15] *z*-coordinates, non-terrain, quasi-uniform latitude-longitude grid and 18-layer vertical geopotential height. The Navier-Stokes primitive equations of shallow atmosphere are applied to describe an adiabatic, frictionless and incompressible model atmosphere. Meanwhile the quasi-Lagrangian time integration is adopted. An ideal case is simulated by using the NCEP reanalysis data as initial value fields. The time is set at 00:00 (UTC, see Fig. 1) on January 10, 2008.

The simulation case is 120 hours of integration with time step of 120 s. A nine-point smoothing with a coefficient of 1/3 is conducted on the entire pressure and temperature fields once every 6 hours in the integration but on the entire level of wind field on every time step. The simulation results (Figs. 2 to 6) show that the cubic-motion advections can simulate both the general circulations and the atmospheric longwave (i.e. Rossby Wave) activities of "trough" and "ridge" in the global Spline Model.

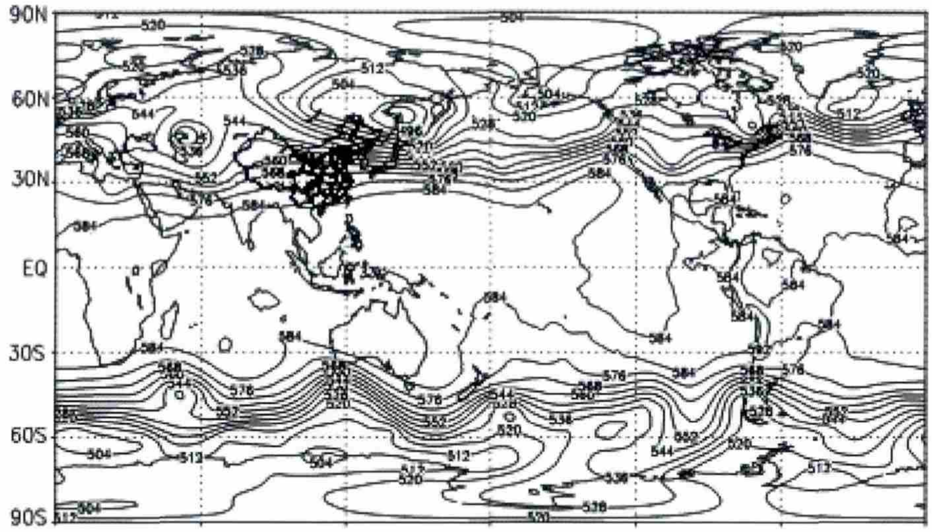


**Figure 1.** Initial geopotential height (dagpm, real line) at 500 hPa at 00:00 January 10, 2008.

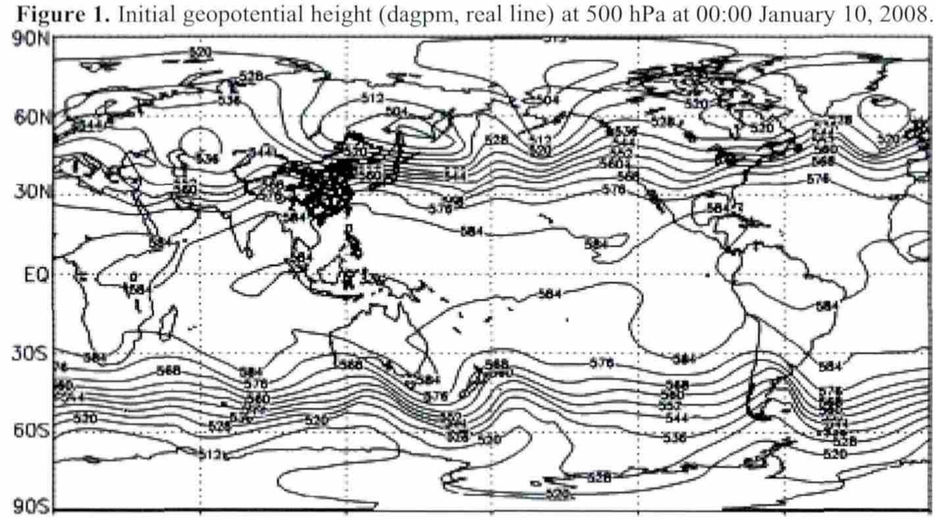


**Figure 2.** Same as Fig.1, but for 24-h forecast geopotential height (dagpm, real line) at 500 hPa.

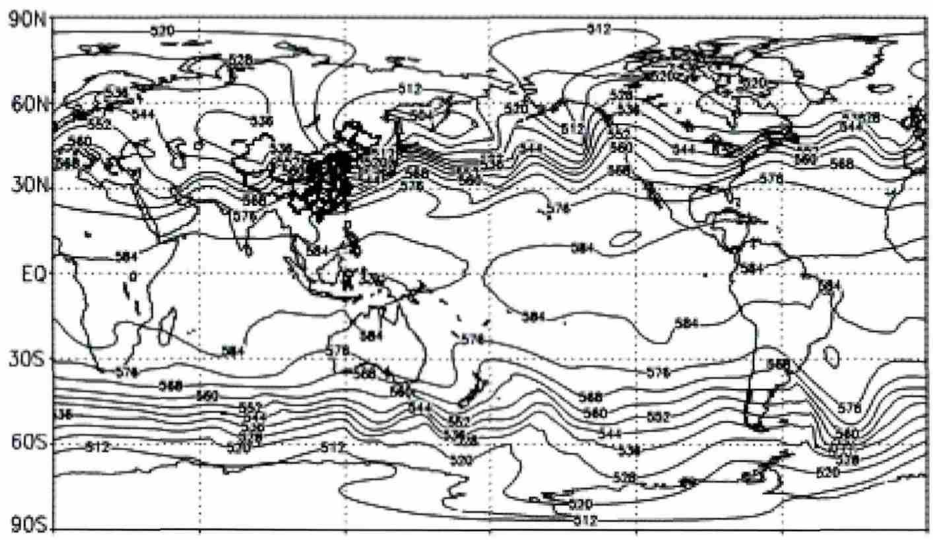


**Figure 3.** Same as Fig.1, but for 48-h forecast geopotential height (dagpm, real line) at 500 hPa.

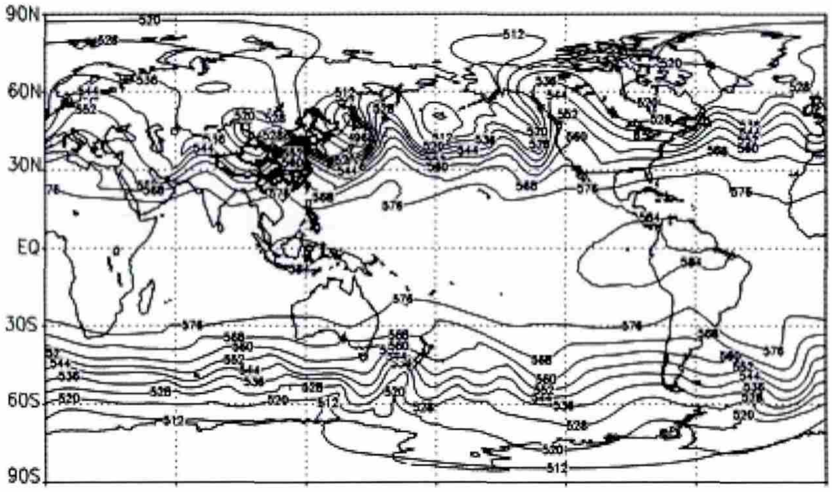


**Figure 4.** Same as Fig.1, but for 72-h forecast geopotential height (dagpm, real line) at 500 hPa.

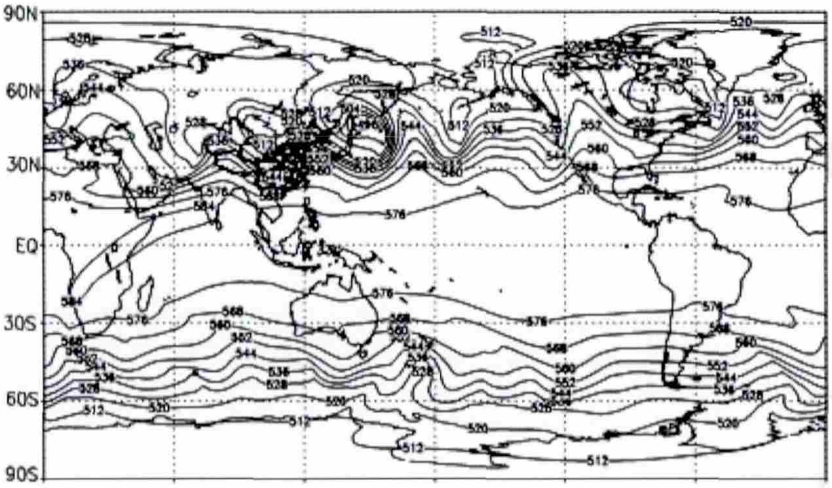


**Figure 5.** Same as Fig.1, but for 96-h forecast geopotential height (dagpm, real line) at 500 hPa.

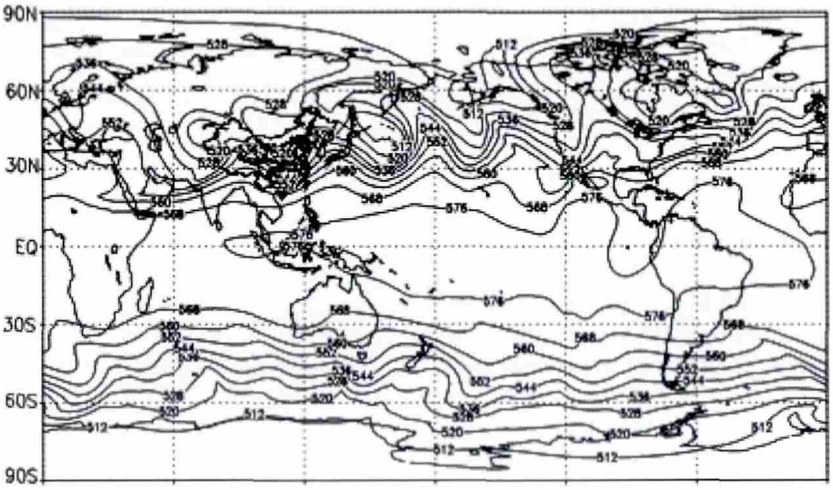


**Figure 6.** same as Fig.1, but for 120-h forecasted geopotential height (dagpm, real line) at 500 hPa.

## 6 SMOOTHING FOR ATMOSPHERIC NON-LINEAR MOTIONS

According to the description of original equations, the essential part of atmosphere motions is non-linear movement, which is neither "linear" nor "cubic". Considering the occurrence of a lot of unstable disturbances on different scales, the interpretation of the cubic movement should be more accurate than the linear one. If no smoothing was conducted in the integration of the above simulation, there are a lot of discontinuities (called "cusp" or "surround") of fitting cubic splines to the pressure and temperature fields, especially to the wind field.

Smoothing global fields for all the variables must distort the model atmospheric movements. Therefore, the smoothed local area or simple point should reflect the real atmospheric movement. Since every time before integration, the determinate curvatures of fitting spline to each variable field can be obtained in advance. We may judge the curvatures for smoothing the unreasonable areas or points conforming to physical interpretation, such as mass conservation for pressure smoothing, energy conservation for temperature smoothing, and momentum conservation for wind smoothing. It is also easier to induce a new 2nd-order derivative bicubic "patch" to the smoothed domain with the Hermite interpolations. If the unstable area was the original place of some synoptic systems (such as a typhoon), however, a nested Spline Model with high resolution would be built in the global model for avoiding smoothing.

## 7 CONCLUSIONS AND DISCUSSIONS

(1) Cubic spline function (spline, bicubic surface and tricubic cube) is featured by the following properties. (a) Contraction law: this spline can converge its 1st-order and 2nd-order with derivatives contracted to the original function. (b) Optimality law: its 2nd-order derivative is optimally approximate to the original function. Thus the cubic spline model is the “optimum” numerical model with 2nd-order differentiable spatial interpolation.

(2) Hermite bicubic patch is a 2nd-order derivative variable field equivalent to perform operation on the mesh. The Coons bicubic curved surface has the 2nd-order differentiable “convergence” and “optimality” with its slopes, curvatures and twists for fitting a variable field. Besides, it has spatial calculation accuracy of 2nd-order differential remainder. The dynamic core of global cubic spline model is the quasi-Lagrangian time-split integration scheme with fitting cubic spline functions for variable fields. The scheme calculates the upstream air parcel, which can be on a global quasi-uniform, latitude-longitude, topological rectangular grid including both polar areas and two poles.

(3) The simulating results of dynamic core of global Spline Model indicates that the model is suitable for the Navier-Stokes primitive equations to preliminarily describe the general circulation and longwave "trough" and "ridge" activities on many Coons bicubic surfaces in terms of the variable fields in the spherical coordinates.

(4) Since the slope and curvature of central difference is derived from cubic spline with a three-point smooth whose smoothing coefficient is 1/3, the central difference approximation always reduces the wave amplitude and slows down its phase velocity in model atmosphere.

(5) In theory, the dynamic core of global Spline Model can be determined by the curvatures of fitting spline to some variable fields. Besides, their curvatures should be judged by smoothing some areas or points to keep the model stable at every time step of the integration

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