THE CISK MECHANISM AND THE INERTIAL GRAVITATIONAL WAVES IN SHEAR FLOW

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ABSTRACT

By using the symmetric equations of atmospheric dynamics in y-z plane with vertical and horizontal shear of wind, the nonlinear ordinary differential equation is derived with the method of travelling wave. Its stability is discussed by using the theory of nonlinear stability and the KDV equation is solved. The effects of linear CISK, nonlinear CISK, inertial stability and vertical shear of wind on the amplitude and the width of isolated inertial gravitational wave are discussed. In order to understand deeply the formation and development of meso-scale synoptic systems, such as the squall line, MCC, the cold surge of Asia high and typhoon, the factors of development of the isolated inertial gravity wave are analysed.

Key words: CISK, nonlinearity, isolated inertial gravitational waves

I. INTRODUCTION

Since Charney and Eliassen (1964) published the theory of the conditional instability of the second kind, it has been applied widely. The effects of CISK on the generation and development of a typhoon in the shear of basic flow were discussed and used to interpret the low frequency oscillation in the tropical atmosphere (Li, 1983a). The influences of CISK upon Rossby waves and Kelvin waves in the low latitude region were illuminated theoretically (Liu et al., 1992) and pointed out that CISK played an important role in the low-frequency oscillation in the tropics. Many problems about CISK were synthesized (Wu, 1987), with the conclusion that using the ageostrophic flow is possibly more reasonable than using the semi-geostrophic flow in the y-direction. All the work above is focused on the fundamental mechanism of linear CISK. Nonlinear CISK was brought forth and considered as an important cause in the sudden development of a typhoon (Li and Zhu, 1989). The different effects between linear and nonlinear CISK on the linear and nonlinear inertial gravitational waves were discussed preliminarily. But the effects of the shear basic flow on linear and nonlinear waves are not involved in the process of CISK. How are the different influences of the shear wind upon linear and nonlinear waves in the process of CISK heating? In this paper, the problem mentioned above is studied.

In the second section of the paper, the basic equations describing the atmospheric motion are given. By using the travelling wave method, a nonlinear ordinary differential equation is derived. In the third section the stability of the linear inertial gravity waves in processes of two types of CISK heating and the shear basic flow on the linear waves are discussed. In the fourth section nonlinear inertial gravity waves and their solutions in CISK process are studied. Effects of the shear wind on the strength and width of nonlinear isolated inertial gravity wave are analyzed. The conclusions are shown in the last section.

Ⅱ. BASIC EQUATIONS

The symmetric, two-dimensional nonlinear equations in y-z plane with Bossinesq approximation are used:

$$\begin{cases} \frac{\partial u}{\partial t} + \bar{u}_z w - (f - \bar{u}_y) v + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0, \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z} + fu = -\frac{\partial \phi}{\partial y}, \\ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial z}\right) + w\frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} - f\bar{u}_z v = Q - N^2 w, \end{cases} \tag{1}$$

where $\bar{u}_y = \frac{\partial \bar{u}}{\partial y}, \bar{u}_z = \frac{\partial \bar{u}}{\partial z}$. Q is CISK heating rate, $\frac{\partial \bar{\phi}}{\partial y} = -f\bar{u}$ is used and the temperature advection is omitted in the thermal equation, so temperature variation is caused by CISK heating, the horizontal and vertical shear of wind u, v, w, ϕ are the disturbances $N^2 = N_0^2 + \frac{\partial^2 \bar{\phi}}{\partial z^2}$, N_0^2 is the static stability parameter in the static atmosphere. The others in Eq. (1) are frequently used in meteorology. Let

$$\begin{aligned} &u = U(\theta),\ v = V(\theta),\ w = \overline{W}(\theta),\ \phi = \Phi(\theta), \\ &\theta = l \cdot y + nz - \omega t, \end{aligned} \tag{2}$$

where l, n are wavenumbers in y-direction and z-direction. ω is the frequency of the inertial gravity wave.

Substituting Eq. (2) into (1), we get

$$\begin{cases} -\omega\dot{U} + (lV + n\overline{W})\dot{U} + \bar{u}_z\overline{W} - (f - \bar{u}_y)V = 0, \\ -\omega\dot{V} + (lV + n\overline{W})\dot{V} + fU = -l\dot{\Phi}, \\ l\dot{V} + n\dot{\overline{W}} = 0, \\ n(n\overline{W} - \omega)\dot{\Phi} = Q - N^2\overline{W} + f\bar{u}_z)V, \end{cases} \tag{3}$$

where $\dot{} = \frac{d}{d\theta}$. Integrating the third equation of Eq. (3) and letting the integration constant being zero, we obtain

$$lV + n\overline{W} = 0. \tag{4}$$

Substituting Eq. (4) into (3), eliminating V and $\dot{V}$

$$\begin{cases} -\omega\dot{U} + \bar{u}_z\overline{W} + (f - \bar{u}_y)\frac{n}{l}\overline{W} = 0, \\ \frac{\omega n}{l}\dot{\overline{W}} + fU = -l\dot{\Phi}, \\ n(n\overline{W} - \omega)\dot{\Phi} + \frac{f\bar{u}_z n\overline{W}}{l} = Q - N^2\overline{W}, \end{cases} \tag{5}$$

Eliminating U in the first and second equations of Eq. (5), solving and getting the formula about Φ, then substituting it into third equation of Eq. (5), we obtain

$$\ddot{\overline{W}} + \frac{1}{n^2\dot{\omega}^2}[f\bar{u}_z ln + n^2 I^2]\overline{W} = -\frac{\dfrac{fnl\bar{u}_z\overline{W} + l^2(N^2\overline{W} - Q)^2}{n^2\dot{\omega}^2}}{\left(1 - \dfrac{n\overline{W}}{\dot{\omega}}\right)}, \qquad (6)$$

where $I^2 = f(f - \bar{u}_y)$ is the inertial stability parameter.

Now we consider two types of CISK heating, one is the linear (it is similar to that of Charney & Eliassen),

$$Q = bN^2\eta\overline{W}. \qquad (7)$$

Charney el al. used N^2 in Eq. (7) without the effects of basic flow. Another is the nonlinear (it is similar to that of Li et al., 1989),

$$Q = N^2\eta(d\overline{W}^2 + e\overline{W}^3), \qquad (8)$$

where η is the CISK heating parameter, b, d, e are constants. b is a non-dimensional number. Dimensions of d, e are s/m, s^2/m. Substituting formulas (7) and (8) into Eq. (6), taking $\eta b = \eta_1, \eta d = \eta_2, \eta e = \eta_3$, then Eq. (6) is written as

$$\ddot{\overline{W}} + \frac{1}{n^2\dot{\omega}^2}[fln\bar{u}_z + n^2 I^2]\overline{W} = -\frac{fln\bar{u}_z\overline{W} + N^2l^2(1 - \eta_1)}{n^2\dot{\omega}^2(1 - n\overline{W}/\dot{\omega})}, \qquad (9)$$

$$\ddot{\overline{W}} + \frac{1}{n^2\dot{\omega}^2}[fln\bar{u}_z + n^2 I^2]\overline{W} = -\frac{fln\bar{u}_z\overline{W} + N^2l^2(\overline{W} - \eta_2\overline{W}^2 - \eta_3\overline{W}^3)}{n^2\dot{\omega}^2(1 - n\overline{W}/\dot{\omega})}. \qquad (10)$$

Eqs. (9) and (10) are basic ones by which effects of the shear wind on linear and nonlinear inertial gravitational waves are studied.

Ⅲ. INFLUENCES OF THE SHEAR BASIC FLOW AND TWO TYPES OF CISK HEATING ON LINEAR INERTIAL GRAVITATIONAL WAVES

Considering $\dfrac{\overline{W}}{(\dot{\omega}/n)} = \dfrac{\overline{W}}{C_z} \ll 1$,

$$\frac{1}{1 - n\overline{W}/\dot{\omega}} = 1 + \frac{n\overline{W}}{\dot{\omega}} + \left(\frac{n\overline{W}}{\dot{\omega}}\right)^2 + \cdots\cdots, \qquad (11)$$

Substituting (11) into (9) and (10) and making Taylor expansion at $\overline{W} = 0$, taking the first-order approximation, we can get

$$\ddot{\overline{W}} + \frac{1}{n^2\dot{\omega}^2}[2fln\bar{u}_z + n^2 I^2 + N^2l^2(1 - \eta_1)]\overline{W} = 0, \qquad (12)$$

$$\ddot{\overline{W}} + \frac{1}{n^2\dot{\omega}^2}[2fnl\bar{u}_z + n^2 I^2 + N^2l^2]\overline{W} = 0. \qquad (13)$$

Comparing Eqs. (12) with (13), we can know that when $\eta_1 = 0$, Eq. (13) is a case of

(12). From this, nonlinear CISK donesn't exert any influence on the stability of linear inertial gravity waves.

The criteria about the stability of inertial gravity waves are:

If $\dot{\omega}^2>0$,

$2fln\bar{u}_z+n^2I^2+N^2l^2\ (1-\eta_1)\geqslant 0$, stable

$2fln\bar{u}_z+n^2I^2+N^2l^2\ (1-\eta_1)<0$. unstable

$N^2<0$ represents the static instable condition. $I^2<0$ is the inertial instable one. $\bar{u}_z$ represents the atmospheric baroclinity. If the vertical shear of wind is constant., when $N^2>0$, the larger η_1 is, the more easily unstable condition is fulfilled; when $N^2<0$, the smaller η_1 is, the more easily instable condition is satisfied. If $\dot{\omega}^2<0$, the conclusions are converse.

Ⅳ. EFFECTS OF THE SHEAR BASIC FLOW AND CISK ON NONLINEAR INERTIAL GRAVITATIONAL WAVES

Substituting Eq. (11) into Eqs. (9) and (10), taking second-order approximation, we can get

$$\ddot{\overline{W}}+\frac{1}{n^2\dot{\omega}^2}[2fnl\bar{u}_z+n^2I^2+N^2l^2(1-\eta_1)]\overline{W}+\frac{1}{n^2\dot{\omega}^2}[fnl\bar{u}_z+N^2l^2(1-\eta_1)]\cdot\frac{n\overline{W}^2}{\dot{\omega}}=0, \tag{14}$$

$$\ddot{\overline{W}}+\frac{1}{n^2\dot{\omega}^2}[2fnl\bar{u}_z+n^2I^2+N^2l^2]\overline{W}+\frac{1}{n^2\dot{\omega}^2}\left[fnl\bar{u}_z\cdot\frac{n\overline{W}^2}{\dot{\omega}}+Nl^2\cdot\frac{n\overline{W}^2}{\dot{\omega}}-N^2l^2\eta_2\overline{W}^2\right]=0, \tag{15}$$

Obviously Eqs. (14) and (15) are KDV ones. Their special solutions are the isolated waves. The stability of which can be tested with the linear stability criteria. For stable isolated inertial gravitational waves, their solutions are:

$$\overline{W}=3G_1sech^2\left(\frac{\sqrt{G_1}/2}{2G_2}\right)-\frac{G_1}{G_2}, \tag{16}$$

$$\overline{W}=3G_3sech^2\left(\frac{\sqrt{G_3}/2}{2G_4}\right)-\frac{G_3}{G_4}, \tag{17}$$

where

$$\begin{cases} G_1=-\dfrac{1}{n^2\dot{\omega}^2}[2fnl\bar{u}_z+n^2I^2+N^2l^2(1-\eta_1)], \\ G_2=-\dfrac{1}{n^2\dot{\omega}^2}[fnl\bar{u}_z+N^2l^2(1-\eta_1)], \\ G_3=-\dfrac{1}{n^2\dot{\omega}^2}[2fnl\bar{u}_z+n^2I^2+N^2l^2], \\ G_4=-\dfrac{1}{n^2\dot{\omega}^2}\left[fnl\bar{u}_z+N^2l^2-N^2l^2\eta_2\dfrac{\dot{\omega}}{n}\right], \\ \theta=ly+nz-\dot{\omega}t. \end{cases}$$

The solutions of the unstable isolated inertial gravity waves are:

$$W = 3G_1 sech^2\left[\frac{\sqrt{G_1}/2}{2G_2}\right], \tag{18}$$

$$W = 3G_3 sech^2\left[\frac{\sqrt{G_3}/2}{2G_4}\right]. \tag{19}$$

So we can solve the amplitudes and widths of the isolated waves as follows:

$$A_1 = \frac{3G_1}{2G_2}, \quad A_2 = \frac{3G_3}{2G_4}, \tag{20}$$

$$\frac{1}{\mu_1} = \left[\frac{\sqrt{G_1}}{2}\right]^{-1}, \quad \frac{1}{\mu_2} = \left[\frac{\sqrt{G_3}}{2}\right]^{-1}. \tag{21}$$

We will discuss the effects of the shear basic flow and CISK on the amplitude and width of the isolated inertial gravity wave as follows.

1. *Effects of linear CISK and the shear flow on the strength of the isolated inertial gravitational waves*

As known from formula (20), we obtain

$$A_1 = \frac{3}{2}\cdot\frac{l}{n}\cdot C_y\left[1 + \frac{1 + \frac{I^2}{f\bar{u}_z}\cdot\frac{n}{l}}{1 + N^2\cdot\frac{l}{n}(1-\eta_1)/f\bar{u}_z}\right], \tag{22}$$

where C_y is the wave velocity along y-direction. According to formula (22), we can conclude that if $f\bar{u}_z > 0$, when $I^2 > 0$, the larger I^2 makes the amplitude of the isolated wave increase; when $I^2 < 0$, the smaller I^2 makes the amplitude reduce. If $N^2(1-\eta_1) > 0$, the smaller it is, the larger the amplitude is; the larger it is, the smaller the amplitude is; *vice versa*. In order to illuminate quantitatively effects of the shear flow and linear CISK on the isolated inertial gravitational waves, letting $\frac{l}{n} \sim 10^{-2}$, $C_y \sim 22\ m/s$, $N^2 \sim 10^{-3}s^{-2}$, $I^2 \sim 10^{-10}s^{-2}$, $\bar{u}_z \sim 10^{-4}s^{-1}$ (Xia and Zheng., 1993), we get Table 1.

Table 1. Relations between the amplitude of the isolated inertial gravity wave A_1 and heating parameter η_1 (unit: m/s).

	0	0.5	1.0	1.5	2.0
A_1	0.36	0.39	0.69	0.24	0.29

As shown in Table 1, $\eta_1 = 1$ is the point of the violent variance. If letting $\eta_1 = 0.5$, we can obtain Table 2 as follows:

Table 2. Relations among the amplitude of the isolated inertial gravity wave A_1 (m/s), the inertial stability parameter $I^2(1/s^2)$ and the vertical shear of wind $\bar{u}_z(1/s)$.

$\bar{u}_z$	I^2				
	-10^{-10}	-10^{-11}	0	10^{-11}	10^{-10}
10^{-3}	0.33	0.38	0.38	0.39	0.44
10^{-4}	0.27	0.33	0.33	0.34	0.40
0	0.26	0.32	0.32	0.34	0.40
-10^{-4}	0.26	0.32	0.32	0.33	0.39
-10^{-3}	0.17	0.24	0.24	0.26	0.33

As known in Table 1, when $N^2 > 0$, the stronger CISK heating is, the larger the amplitude of the isolated wave is; the weaker CISK heating is, the smaller the amplitude of the isolated wave is. In a case of $N^2 < 0$, there is a contrary situation.

As shown in Table 2, while the inertial stability parameter is from the unstatble to the stable, the isolated wave amplitude increases, the more stable the inertial is, the larger the amplitude is; while the parameter is from the stable to the instable, the amplitude decreases, the more unstable the inertial is, the smaller the amplitude is.

When $\bar{u}_z$ varies from the negative to the positive, the amplitude of the isolated wave increases; on the contrary, the amplitude decreases. From this we can infer that when the easterly is at the low level and the westerly at the high level, the isolated inertial gravity wave with a large amplitude will appear; when the westerly is at the low level and the easterly is at the high level, the isolated wave with a small amplitude will generate.

2. *Effects of the shear flow and nonlinear CISK on the isolated inertial gravitational wave*

From Eq. (20) the amplitude of the isolated inertial gravitational wave is derived as follows:

$$A_2 = \frac{3}{2} \cdot \frac{l}{n} \cdot C_y \left[\frac{1 + \dfrac{fnl\bar{u}_z + n^2 I^2}{fnl\bar{u}_z + N^2 l^2}}{1 - \dfrac{N^2 l^2 \eta_2 C_z}{fnl\bar{u}_z + N^2 l^2}} \right], \tag{23}$$

where $C_z = \frac{\hat{\omega}}{n}$ is the wave velocity in z-direction. For convenience sake, we let the atmosphere be the static stable ($N^2 > 0$). While $f\bar{u}_z > 0$, the larger I^2 makes the nonlinear isolated wave intensify; the smaller I^2 makes the nonlinear isolated wave decay. When $f\bar{u}_z < 0$, there is a contrary situation. Taking the parameters' values same as ones in Table 1, the relations can be estimated quantitatively as follows.

Table 3. Relations among the amplitude of the nonlinear isolated inertial gravity wave A_2 (m/s), the heating parameter η_2 and the wave vertical velocity C_z (m/s).

C_z	η^2				
	0	0.5	1.0	1.5	2.0
22	0.39	0.43	0.49	0.56	0.64
-22	0.39	0.35	0.32	0.30	0.28

Taking $\eta_2=0.5$ and other variations same as ones in Table 2, we can get Table 4.

Table 4. Relations among of the amplitude of the isolated inertial gravity wave A_2 (m/s), the inertial stability parameter I^2 (s^{-2}) and the vertical shear of wind $\bar{U}_z$ (1/s).

$\bar{u}_z$	I^2		
	-10^{-10}	0	10^{-10}
10^{-3}	0.37	0.40	0.43
0	0.33	0.37	0.41
-10^{-3}	0.29	0.33	0.38

As known in Table 3, when the wave propagates upwards ($C_z>0$), the stronger (weaker) nonlinear CISK is, the larger (smaller) an amplitude of the isolated wave is; when the wave propagates downwards ($C_z<0$), the results are converse.

From Table 4, we can obtain that in the process of nonlinear CISK heating the effects of the shear basic flow on an amplitude of the isolated wave are similar to the results shown in Table 2.

3. *Effects of linear CISK heating on the width of the isolated inertial gravitational wave*

The width of the isolated inertial gravitational wave is got according to Eq. (21).

$$\frac{1}{\mu_1}=\frac{2}{\sqrt{G_1}}=2C_y\left\{l^2/\left[2fnl\bar{u}_z\frac{l}{n}+I^2+N^2l^2\cdot(1-\eta_1)/n^2\right]\right\}^{1/2}. \tag{24}$$

Given values to some variations in formula (24), the relations can be estimated quantitatively and shown in Tables 5 and 6 as follows.

Table 5. Relations between CISK heating parameter η_1 and the width of the isolated inertial gravity wave (unit: km).

η_1	0	0.5	1.0
$1/\mu_1$	13.3	16.6	31.3

Table 6. Relations among the width of the isolated inertial gravity wave $1/\mu_1$ (km), the inertial stability parameter I^2 (s^{-2}) and the vertical shear of wind $\bar{u}_z$ (l/s).

$\bar{u}_z$	I^2		
	-10^{-10}	0	10^{-10}
10^{-3}	18.0	16.6	15.6
0	22.0	19.7	18.0
-10^{-3}	31.1	25.4	22.0

As shown in Tables 5 and 6, if $N^2>0$, the stronger linear CISK heating is, the wider the width of the isolated wave is; the weaker the heating is, the less narrow the width is. If $N^2<0$, the conclusions are contrary.

While the inertial stability is from the stable to the unstable, the width of the isolated wave widens, *vice versa*. The vertical shear of wind is from the positive to negative, the width of the isolated wave widens; on the contrary, the width narrows.

4. *Effects of the shear flow and nonlinear CISK heating on the width of isolated inertial gravitational wave*

As known in Eq. (4), nonlinear CISK heating doesn't affect the width of isolated inertial gravitational wave. Effects of the shear basic flow on the width of the nonlinear isolated wave are similar to the ones discussed above.

According to the analyses above, we conclude that N^2, I^2, η_1, η_2 and $\bar{u}_z$ are the physical factors which cause the variation of the isolated inertial gravity wave; N^2, I^2, η_1 and $\bar{u}_z$ are the ones which result in the variation of the width of the isolated wave. In general, squall line, the cold surge ahead of Asia high, MCC, and the heavy rain are associated closely with the isolated inertial gravity wave. Therefore, according to analyses above, $I^2 > 0$ is in the left of the low level jet but I^2 in the right of the low level jet is smaller than the one in the left or is the negative, so the strength of the isolated inertial gravity wave in the left of the low level jet is stronger than that in the right. This provides a physical illumination that more rain gush happened in the left of the low level jet.

As shown in Tables 2 and 4, when given CISK heating and the inertial stability parameter I^2, a synoptic system happening in the condition of $\bar{u}_z > 0$ is stronger than the one under the condition of $\bar{u}_z < 0$. This is similar to the result discussed in the literature (Xia et al., 1993).

V. CONCLUSION

For simplicity, the symmetric, two-dimensional equations in y-z plane with Boussinesq approximation are used in this paper. Applying the travelling wave method, the CISK inertial gravitational waves in shear flow are discussed.

In a process of linear CISK, when $\eta_1 = 1.0$ the amplitude of the isolated inertial gravitational waves is the largest with η_1 increasing from 0 to 2.0 (shown in Table 1) and the width of the isolated wave widens too (shown in Table 5). In a process of nonlinear CISK, taken $C_y = 22$ m/s, when η_1 increases, the strength of the isolated wave intensifies but the width of the isolated wave doesn't vary.

While the shear basic flow is given, in a case of the inertial stable condition, the amplitude of the isolated wave is larger than the one in a case of the static unstable condition, so is the width of the isolated wave.

Under the condition of CISK heating and I^2 being constant, when the vertical shear of wind $\bar{u}_z$ is positive, the strength of the isolated wave is strong and the width of the isolated wave is wide; *vice versa*.

The conclusions above are analysed theoretically. They are coincided with some reality. But they need to be tested and studied further by more observational data.

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