CHAOTIC OUTPUT OF SST FLUCTUATION STOCHASTIC MODEL WITH GIVEN PARAMETERS①

Yan Shaojin (严绍瑾) and Peng Yongqing (彭永清)

Nanjing Institute of Meteorology, Nanjing 210044

Received 27 July 1994, accepted 6 January 1995

ABSTRACT

Starting from the Saltzman's air-sea stochastic climatic model, we have derived a langevin-type equation describing SST fluctuation and the related Fokker-Plank expression, which were then numerically solved with parameters given, yielding the probability density curve $P\ (x,\ t)$ of multiple bifurcations, with the Cantor set of images given in phase space of $P(x,t)$ and $P(x,t+\tau)$, thereby indicating that chaotic output comes from the random system under the conditions of the above parameters.

Key words: stochastic system, multiple bifurcation, Cantor set

Ⅰ. INTRODUCTION

In his numerical study of thermal convection, Lorenz (1963) discovered an entirely deterministic system of third-order ordinary differential equations, which can give seemingly disordered aperiodic output over a given range of parameter values, referred to by himself as "deterministic aperiodic flow". Subsequently, numerous studies show that as long as the deterministic equations become more or less complicated, "random" behaviors are likely to take place in the system, which is the intrinsic stochasticity typical of a dynamic system, and actually the result of extremely high sensitivity of the system to initial conditions. Deepened exploration of chaos mechanisms indicates that the chaos associated with the intrinsic random property differs utterly from that of common random phenomena. Such stochasticity is not responsible for irregular behaviors but orderly ones. This orderly property gives rise to a countless number of embedded autosimilar geometric structures and is marked by universality.

This issue has recently been a growing concern. Are chaotic behaviors displayed in a dynamic system produced merely by a complicated deterministic system? Is a random system possibly responsible for chaotic output? Under what conditions can such output be obtained? They are no doubt of great importance to climatological research. About a decade ago, in dealing with air-sea interaction, Saltzman, by considering the stochastic term of the original dynamic-climatic model, discovered that as this term grows, the periodic solution of the deterministic dynamic system becomes disintegrated step by step so as to form a wide-band frequency spectrum, a result that needs to be further addressed. The aim of the present paper is to investigate if there are probably any chaotic behaviors by the system in the context of the Saltzman model.

Ⅱ. BASIC RELATIONS DESCRIBING CHAOTIC BEHAVIORS OF A RANDOM SYSTEM

The equation for a random system can be given as

① This study is supported by the National Natural Sciences Foundation of China.

$$\frac{dx_i}{dt} = f_i(x_1, \cdots\cdots, x_n) + \eta_i(t) \qquad i = 1, 2, \cdots\cdots, n, \tag{1}$$

where η_i represents the stochastic term . Following the theroy on random processes , if η_i meets the condition

$$\langle \eta_i(t), \eta_j(t) \rangle = D_{ij}\delta(t), \tag{2}$$

with δ denoting the Dirac function and D_{ij} the correlated deviation, then the Fokker-Plank equation in relation to (1) takes the form

$$\frac{\partial p}{\partial t} + \sum_i^n \frac{\partial}{\partial x_i}(f_i, p) - \frac{1}{2}\sum_i^n \sum_j^n \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} p = 0, \tag{3}$$

where P is the probability density function. (3) is also called the principal equation for the evolution of probability density. Eqs. (1) and (3) are normally the basic relations in dealing with chaotic behaviors of a random system. Theoretical study (Kapitaniak, 1988) shows that for chaotic behaviors of a random system, one of the important conditions is that with certain parameter values available, the probability density curve has multiple maxima, i. e. , multiple bifurcations. Next, in view of the fact that the probability desity function $P\ (x,\ t)$ is a temporal function, the determination of the presence of chaotic behaviors in the associated system depends on more than the existence of multiple bifurcations on the probability density curve, i. e. , it is needed to construct a phase space with $P\ (x, t)$ and $P\ (x, t+\tau)$ as coordinates. The occurrence of Cantor set image therein for certain parameter magnitudes serves as the important forerunner of such behaviors.

Ⅲ. A STOCHASTIC CLIMATIC SYSTEM

Saltzman (1978) proposed an atmosphere-sea-ice cover incorporated climatic model, as illustrated in Fig. 1.

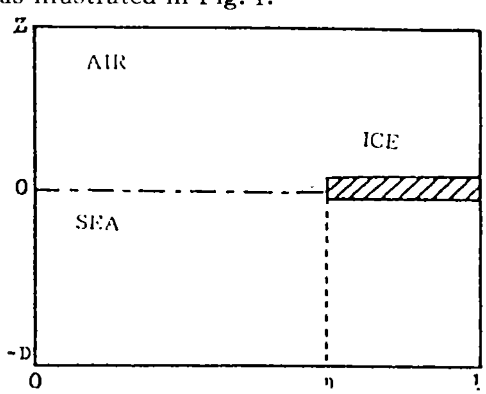

Fig. 1. Schematic of Saltzman simple climate model.

The model takes the form

$$\begin{cases} \dfrac{d\eta'}{dt} = \varphi_1\theta' - \varphi_2\eta', \\ \dfrac{d\theta'}{dt} = -\varphi_1\eta' + \varphi_2\theta' - \varphi_3\eta'^2\theta', \end{cases} \tag{4}$$

where $\eta = \sin\zeta$, with ζ as the latitude for the limit to ice cover; $\theta = \frac{1}{D}\int_0^1\int_0^D T\, dz\, d\zeta$ denotes the mean SST at the ice limit with D as the sea depth; $\eta' = \eta - \bar{\eta}$, $\theta' = \theta - \bar{\theta}$ and φ_1、φ_2、ψ_1、ψ_2、ψ_3 are the given parameters. Eq. (4) has periodic solutions that constitute a stable limit cycle in the $\eta' - \theta'$ phase space. The climatic model possesses a 1260-year period. It is evident from (4) that the system represents a self-excited oscillation system.

Addition of the random terms f and g into the system leads to

$$\begin{cases} \dfrac{d\eta'}{dt} = \varphi_1\theta' - \varphi_2\eta' + f, \\ \dfrac{d\theta'}{dt} = -\psi_1\eta' + \psi_2\theta' - \psi_3\eta'^2\theta' + g. \end{cases} \tag{5}$$

And with increased random disturbance, the limit cycle in the original phase space gets disintegrated little by little and the periodic solution brings about a wide-band frequency spectrum.

If

$$\eta' = \frac{\varphi_1}{\varphi_2}\theta' \tag{6}$$

is satisfied, then (5) develops into the one-dimensional form (Saltzman, 1988)

$$\frac{d\theta'}{dt} = A\theta' - B\theta'^3 + R, \tag{7}$$

where $A = (\psi_2 - \psi_1\varphi_1\varphi_2^{-1})$; $B = \psi_3\varphi_1^2\varphi_2^{-2}$; R=random term; θ' =SST fluctuation. (7) is the Langevin equation of θ'. By denoting $x=\theta'$, (7) is rewritten as

$$\frac{dx}{dt} = Ax - Bx^3 + R. \tag{8}$$

When the rhs random term meets

$$\langle R(t), R(t') \rangle = \frac{\varepsilon}{2}\delta(t - t'), \tag{9}$$

theoretically, (8) has the principal equation (F-P expression) for its related probability density in the form

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[(Ax - Bx^3)p(x,t)] + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}p(x,t) \quad , \tag{10}$$

where $p(x, t)$ is the function of probability density distribution.

Eq. (10) has a steady state solution of the form

$$p(x) = N^{-1}\exp[-2U(x)/\varepsilon], \tag{11}$$

where N^{-1} is a normalized constant, and

$$U(x) = -\int_{x_0}^{x}(Ax - Bx^3)dx. \tag{12}$$

Next, the time-dependent solution of (10) (see Hu, 1985) is given as

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\varepsilon\sigma(t)}}\exp\left\{-\frac{[x - y(t)]^2}{2\varepsilon\sigma(t)}\right\}\exp\left(\frac{-V^2A}{4}\right)dV, \tag{13}$$

where $y(t)$ has a form

$$\int_{y_0}^{y}\frac{1}{Ay - By^3}dy = t - t_s, \quad t_s = -\frac{1}{3}\ln\varepsilon, \tag{14}$$

and

$$\sigma(t) = (Ay^2 - By^6)\int_{y_0}^{y}\frac{1}{(Ay - By^3)^3}dy. \tag{15}$$

IV. CALCULATIONS

Figs. 2, 3, 4 and 5 are based on numerical calculations through (13), (14) and (15), with plot (a) denoting the probability density curve under the conditions of given parameter magnitudes and plot (b) the Cantor set in the phase space of $p(x, t)$ and $p(x, t+\tau)$.

It is seen from these figures that with the values given, $p(x, t)$ shows multiple bifurcations and the Cantor set is displayed in the defined phase space, suggesting the occurrence of chaotic behaviors shown by the random system (7) for the parameter magnitudes.

V. CONCLUDING REMARKS AND DISCUSSION

a. Starting from the Saltzman's air-sea random model, the original model is con-

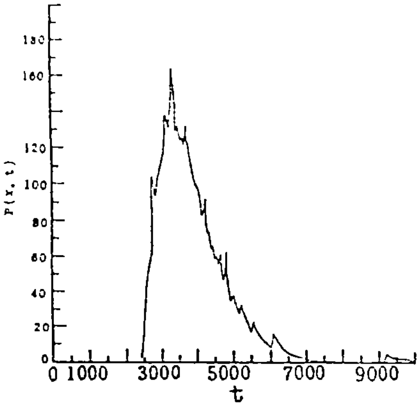

Fig. 2. Probability density function $p(x,t)$ with
$A=4.0\times10^{-11}s^{-1}$,
$B=8.0\times10^{-9}s^{-1}k^{-2}$,
$\varepsilon=6.0\times10^{-11}$, $x_1=1.5$.

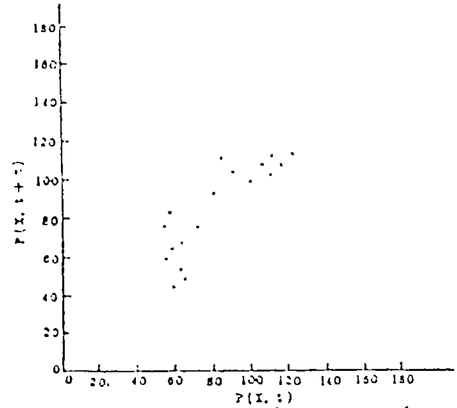

Fig. 3. Cantor pattern in phase space of
$p(x, t) - p(x, t+\tau)$
with $\tau=50$ years.

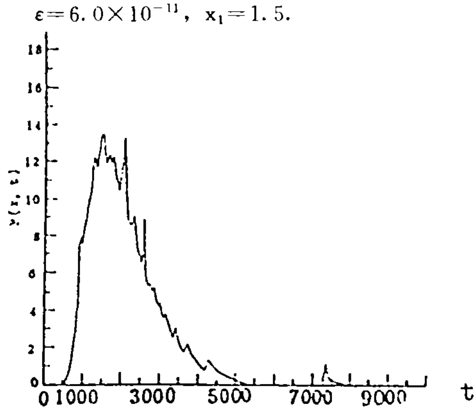

Fig. 4. Probability density curve $p(x,t)$ where
$A=4.0\times10^{-11}s^{-1}$,
$B=8.0\times10^{-9}s^{-1}k^{-2}$,
$\varepsilon=6.0\times10^{-13}$ and $x=-1.5$.

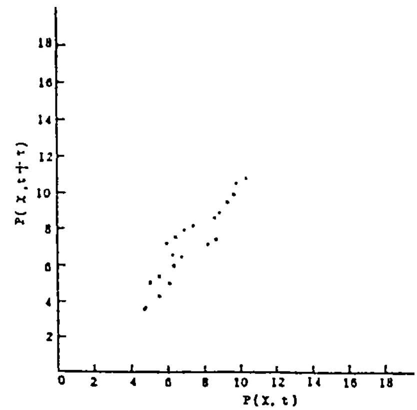

Fig. 5. As in Fig. 3 but for
$p(x_2,t)-p(x_2,t+\tau)$.

verted to the Langevin equation of SST fluctuation θ' with the condition $\eta' = \frac{\varphi_1}{\varphi_2}\theta'$. This condition when substituted into (13), gives $\frac{d\eta'}{dt} = 0$, indicating that the problem addressed in this study is concerned with the chaotic state of random output related to SST fluctuation θ', when η' is steady. Results show that with $A = 4.0 \times 10^{-11}s^{-1}$and $B = 8.0 \times 10^{-9}s^{-1}k^{-2}$available, the stochastic effect $\varepsilon = 6.0 \times 10^{-11}$or $\varepsilon = 6.0 \times 10^{-13}$causes multiple bifurcations on the curve of probability density and the $p(x,t) - p(x,t+\tau)$ phase space displays Cantor set, thereby meeting the needs of chaotic output from the random model developed by Kapitaniak. Under the action of external random force θ' will exhibit chaotic behaviors.

b. For (5) we employ the magnitudes of the coefficients as

$$\varphi_1 = 4 \times 10^{-12}k^{-1}s^{-1},\ \varphi_2 = 5 \times 10^{-11}s^{-1},\ \psi_1 = 4 \times 10^{-9}ks^{-1},$$
$$\psi_2 = 2 \times 10^{-10}s^{-1},\ \psi_3 = 5 \times 10^{-7}s^{-1},$$

with φ_1 and φ_2 as the comprehensive parameters in association with the release of latent heat flux; ψ_1 and ψ_2 as the coefficient of ice-limit thermal inertia; ψ_3 as the quantity of nonlinear vigor.

Change is made of the original parameters for calculation as follows:

$$\varphi_1 = 4 \times 10^{-12} k^{-1} s^{-1},\ \varphi_2 = 4 \times 10^{-10} s^{-1},\ \psi_1 = 4 \times 10^{-9} ks^{-1},$$

$$\psi_2 = 8 \times 10^{-11} s^{-1},\ \psi_3 = 8 \times 10^{-5} s^{-1}.$$

Such change made is of great significance and makes the calculation useful. The ψ_2-dominated positive feedback is reduced; φ_2-controlled ice thermal dissipative effect is increased; ice inertia is intensified so that the sea inertia is made closer to reality. Of course, ψ_3 is strengthened to make the nonlinear effect salient.

c. Research also indicates that with increased nonlinear effect, very weak random disturbance will cause the Cantor structure in the $p(x,t) - p(x,t+\tau)$ phase space in relation to the function of probability density distribution, thus leading to the chaotic output from the stochastic system, an aspect that deserves emphasis. For the original equation (4), although containing a 1260-year periodic solution, it is able to make the frequency band widened under the influence of random force, an aspect that needs to be further replenished. Beyond that, attention should be drawn to the salient impact of the nonlinear effect. And the θ' evolution (hence η') is marked by aperiodicity.

d. The criterion of SST anomaly in current use is based mainly on El Nino phenomnon. As SST fluctuation exceeds 1℃ the anomaly will occur. $x = \pm 1.5$ (implying SST fluctuation $\theta' = \pm 1.5$℃ addressed in this paper) actually takes into account the influence of the anomaly, suggesting that the obtained results represent the chaotic output with SST anomaly available.

REFERENCES

Hu Gang, 1985. On nonsteady state solutions of Fokker—Plank equation of nonlinear drift. *J. Physics*, **34** (5): 573-579.

Kapitaniak, 1988. Chaos in systems with noise. Singapore World Scientific Publishing, 25-36.

Saltzman, B. , 1981. Structural stochastic stability of a simple auto-oscillatory climatic feedback system. *J. Atmos. Sci.* , **38**: 494-503.

Saltzman, B. , 1984. Long period free oscillations in a three-component climate mode. New Perspective in Climate Modelling, pp 293-298.